



TITLE:

CONSTRUCTING LOW-DISCREPANCY SEQUENCES BY USING β -ADIC TRANSFORMATIONS

AUTHOR(S):

NINOMIYA, SYOITI

CITATION:

NINOMIYA, SYOITI. CONSTRUCTING LOW-DISCREPANCY SEQUENCES BY USING β -ADIC TRANSFORMATIONS. 数理解析研究所講究録 1997, 1011: 64-76

ISSUE DATE:

1997-08

URL:

<http://hdl.handle.net/2433/61526>

RIGHT:

CONSTRUCTING LOW-DISCREPANCY SEQUENCES BY USING β -ADIC TRANSFORMATIONS

SYOITI NINOMIYA

IBM Research, Tokyo Research Laboratory

Abstract. A new class of low-discrepancy sequences is constructed by the use of β -adic transformations. Here, β is a real number greater than 1. When β is an integer greater than 2, this sequence becomes the generalized van der Corput sequence in base β . It is also shown that for some special β , the discrepancy of this sequence decreases in the fastest order.

0. Introduction and background

First, we recall the notions of a uniformly distributed sequence and the discrepancy of points ([Niederreiter 1]). A sequence x_1, x_2, \dots in the s -dimensional unit cube $I^s = \prod_{i=1}^s [0, 1)$ is said to be uniformly distributed in I^s when

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N c_J(x_n) = \lambda_s(J)$$

holds for all subintervals $J \in I^s$, where c_J is the characteristic function of J , and λ_s is the s -dimensional Lebesgue measure. If $x_1, x_2, \dots \in I^s$ is a uniformly distributed sequence, the formula

$$(0.1) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x_n) = \int_{I^s} f(x) dx$$

holds for any Riemann integrable function on I^s . The discrepancy of the point set $P = \{x_1, x_2, \dots, x_N\}$ in I^s is defined as follows:

$$(0.2) \quad D_N(\mathcal{B}; P) = \sup_{B \in \mathcal{B}} \left| \frac{A(B; P)}{N} - \lambda_s(B) \right|$$

where $\mathcal{B} \subset \wp(I^s)$ is a non-empty family of Lebesgue measurable subsets and $A(B; P)$ is the counting function that indicates the number of n , where $1 \leq n \leq N$, for which $x_n \in B$. When $\mathcal{J}^* = \{\prod_{i=1}^s [0, u_i), 0 \leq u_i < 1\}$, the star discrepancy $D_N^*(P)$ is defined by $D_N^*(P) = D_N(\mathcal{J}^*; P)$. When $S = \{x_1, x_2, \dots\}$ is a sequence in I^s , we

Key words and phrases. β -adic transformation, discrepancy, ergodic theory, numerical integration, van der Corput sequence.

define $D_N^*(S)$ as $D_N^*(S_N)$, where S_N is the point set $\{x_1, x_2, \dots, x_N\}$. Let S be a sequence in I^s . It is known that the following two conditions are equivalent:

- (a) S is uniformly distributed in I^s ;
- (b) $\lim_{N \rightarrow \infty} D_N^*(S) = 0$.

The following classical theorem shows the importance of the notion of discrepancy.

Theorem 0.1 (Koksma-Hlawka)[1]. *If f has bounded variation $V(f)$ on \bar{I}^s in the sense of Hardy and Krause, then for any $x_1, x_2, \dots, x_N \in I^s$, we have*

$$\left| \frac{1}{N} \sum_{n=1}^M f(x_n) - \int_{I^s} f(x) dx \right| \leq V(f) D_N^*(x_1, \dots, x_N).$$

Schmidt [4] showed that, when $s = 1, 2$, there exists a positive constant C that depends only on s , and the following inequality holds for an arbitrary point set P consisting of N elements:

$$(0.3) \quad D_N^*(P) \geq C \frac{(\log N)^{s-1}}{N}.$$

If (0.3) holds, then there exists a positive constant C that depends only on s , and any sequence $S \subset I^s$ satisfies

$$(0.4) \quad D_N^*(S) \geq C \frac{(\log N)^s}{N}$$

for infinitely many N . Taking account of (0.3) and (0.4), we define a low-discrepancy sequence for the one-dimensional case as follows:

Definition 0.1. Let S be an one-dimensional sequence in $[0, 1)$. If S satisfies

$$\overline{\lim}_{N \rightarrow \infty} \frac{N D_N^*(S)}{\log N} = C \text{ (const),}$$

then S is called a low-discrepancy sequence.

Hereafter we consider only the case where $s = 1$. We now introduce the classical van der Corput sequence [1].

Definition 0.2. Let $p \geq 2$ be an integer. Every integer $n \geq 0$ has a unique digit expansion

$$n = \sum_{j=0}^{\infty} a_j(n) p^j, \quad a_j(n) \in \{0, 1, \dots, p-1\} \text{ for all } j \geq 0,$$

in base p . Then, the radical-inverse function ϕ_p is defined by

$$\phi_p(n) = \sum_{j=0}^{\infty} \tau_j(a_j(n)) p^{-j-1} \quad \text{for all integers } n \geq 0,$$

where τ_j is a permutation of $\{0, 1, \dots, p-1\}$. The van der Corput sequence in base p is the sequence $V_p = \{\phi_p(n)\}_{n=0}^{\infty} \subset [0, 1)$.

Theorem 0.2 [1]. For an arbitrary integer $p \geq 2$, V_p is a low-discrepancy sequence.

In the following part of this paper, the author defines a class of sequences by the use of β -adic transformation ([Rény 3], [Parry 2]) and shows that any member of this class is a low-discrepancy sequence when $\beta = (L + \sqrt{L^2 + 4K})/2$, where L and K are integers greater than 1 and satisfy $K \leq L$. When β is an integer greater than 2, the sequence becomes V_β .

1. β -adic transformation

In this section we define the fibred system and the β -adic transformation, following [Schweiger 5] and [Takahashi 6].

\mathbf{R} , \mathbf{Z} , and \mathbf{N} are the sets of all real numbers, all integers, and all natural numbers, respectively. For $x \in \mathbf{R}$, $[x]$ denotes the integer part of x .

Definition 1.1. Let B be a set and $T : B \rightarrow B$ be a map. The pair (B, T) is called a fibred system if the following conditions are satisfied:

- (a) There is a finite countable set A .
- (b) There is a map $k : B \rightarrow A$, and the sets

$$B(i) = k^{-1}(\{i\}) = \{x \in B : k(x) = i\}$$

form a partition of B .

- (c) For an arbitrary $i \in A$, $T|_{B(i)}$ is injective.

Definition 1.2. Let $\Omega = A^{\mathbf{N}}$ and $\sigma : \Omega \rightarrow \Omega$ be the one-sided shift operator. Let $k_j(x) = k(T^{j-1}x)$. We derive a canonical map $\varphi : B \rightarrow \Omega$ from

$$\varphi(x) = (k_j(x))_{n=1}^{\infty}.$$

φ is called the representation map.

We have the following commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{T} & B \\ \varphi \downarrow & & \downarrow \varphi \\ \Omega & \xrightarrow{\sigma} & \Omega \end{array}$$

Definition 1.3. If a representation map φ is injective, φ is called a valid representation.

Definition 1.4. Let $\omega \in \Omega$. If $\omega \in \text{Im}(\varphi)$, ω is called an admissible sequence.

Definition 1.5. The cylinder of rank n defined by $a_1, a_2, \dots, a_n \in A$ is the set

$$B(a_1, a_2, \dots, a_n) = B(a_1) \cap T^{-1}B(a_2) \cap \dots \cap T^{-n+1}B(a_n).$$

We define B to be a cylinder of rank 0.

Definition 1.6. Let $\beta > 1$ and $\beta \in \mathbf{R}$. Let $f_\beta : [0, 1) \rightarrow [0, 1)$ be a function defined by

$$f_\beta(x) = \beta x - [\beta x].$$

Let $A = \mathbf{Z} \cap [0, \beta)$. Then, we have the following fibred system $([0, 1), f_\beta)$:

$$(1.1) \quad \begin{array}{ccc} [0, 1) & \xrightarrow{f_\beta} & [0, 1) \\ \varphi \downarrow & & \downarrow \varphi \\ \Omega & \xrightarrow{\sigma} & \Omega \end{array}$$

The representation map φ of this fibred system is defined by

$$x = \sum_{n=0}^{\infty} \frac{a_n}{\beta^{n+1}} \iff \varphi(x) = (a_0, a_1, \dots, a_n, \dots) \in \Omega.$$

This fibred system $([0, 1), f_\beta)$ is called a β -adic transformation. In this situation, we define $\zeta_\beta \in \Omega$ by

$$(1.2) \quad \zeta_\beta = \lim_{x \nearrow 1} \varphi(x).$$

We also define $X_\beta \subset \Omega$ to be the set of all admissible sequences.

For a sequence $a \in \Omega$, we write the i -th element of a as $a(i)$, that is, $a = (a(1), a(2), \dots)$. We remark that φ is not a valid representation at this point, because $(a_1, a_2, \dots, a_n, 0, 0, \dots)$ and $(a_1, a_2, \dots, a_n - 1, \zeta_\beta(1), \zeta_\beta(2), \dots)$ are two different representations of the same $x = \sum_{i=1}^n a_i \beta^{-i}$. In this paper we adopt the former representation and make φ valid. We derive the following propositions directly from this definition.

Proposition 1.1.

$$X_\beta = \{\omega \in \Omega \mid \forall n \in \mathbf{Z}_{\geq 0} \quad \sigma^n \omega \prec \zeta_\beta\},$$

where $\omega \prec \psi$ means that ω precedes ψ in lexicographical order.

Proposition 1.2. For an arbitrary $i \in A$,

$$B(i) = \begin{cases} [\frac{i}{\beta}, \frac{i+1}{\beta}), & 0 \leq i < [\beta] \\ [\frac{[\beta]}{\beta}, 1), & \text{otherwise} \end{cases}$$

holds.

Let $\rho_\beta(x) = \sum_{n=0}^{\infty} a_n \beta^{-n-1}$; then, we have

$$\rho_\beta(X_\beta) = [0, 1]$$

and the following commutative diagram:

$$(1.3) \quad \begin{array}{ccc} [0, 1) & \xrightarrow{f_\beta} & [0, 1) \\ \varphi \downarrow \quad \uparrow \rho_\beta & & \rho_\beta \uparrow \quad \downarrow \varphi \\ \Omega & \xrightarrow{\sigma} & \Omega \end{array}$$

2. Constructing the sequence

In this section, a sequence $N_\beta \subset [0, 1]$ is defined by the use of β -adic transformation. Let $\beta \in \mathbf{R}_{>1}$ and let $([0, 1], f_\beta)$ be a fibred system (1.3). Let $B = [0, 1]$, and $A, \Omega, (X_\beta, \sigma), \rho_\beta, \varphi, \zeta_\beta, B(a_1, \dots, a_n)$ be the same as in the previous section.

Definition 2.1. For an arbitrary $n \in \mathbf{Z}_{\geq 0}$, $X_\beta(n), Y_\beta(n) \subset X_\beta$, $F_\beta(n) \in \mathbf{Z}$, and $G_\beta(n) \in \mathbf{Z}$ are defined as follows:

$$\begin{aligned} X_\beta(n) &= \begin{cases} \{(0, 0, \dots)\}, & n = 0 \\ \{\omega \in X_\beta \mid \sigma^{n-1}\omega \neq (0, 0, \dots) \text{ and } \sigma^n\omega = (0, 0, \dots)\}, & n \neq 0 \end{cases} \\ Y_\beta(n) &= \cup_{i=0}^n X_\beta(i) \\ F_\beta(n) &= \#X_\beta(n) \\ G_\beta(n) &= \sum_{i=0}^n F_\beta(i) = \#Y_\beta(n) \end{aligned}$$

It is apparent that

$$F_\beta(n) \leq ([\beta] + 1)^{n-1}.$$

Definition 2.2. For an arbitrary $n \in \mathbf{N}$, define $l_n \in \mathbf{N}$ to satisfy $G_\beta(l_n) < n \leq G_\beta(l_n + 1)$. Define $\tau_n : X_\beta(n) \rightarrow \oplus_{i=1}^n A$ by $\tau_n((k_1, \dots, k_n)) = (k_n, \dots, k_1)$. Induce the right-to-left lexicographical or reverse right-to-left lexicographical order to $X_\beta(l_n + 1) = \{\omega_1, \omega_2, \dots, \omega_{F_\beta(l_n + 1)}\}$; that is to say, for all $i < j$, $\tau_n(\omega_i) \prec \tau_n(\omega_j)$ or $\tau_n(\omega_j) \prec \tau_n(\omega_i)$ holds, respectively. In this situation, the sequence N_β is defined as follows:

$$N_\beta = \{\rho_\beta(\omega_{n-l_n})\}_{n=1}^\infty$$

In this paper, we assume that the elements of $X_\beta(l_n + 1)$ are arranged in right-to-left lexicographical order.

From this definition, we immediately have the following proposition:

Proposition 2.1. If $\beta \in \mathbf{Z}_{\geq 2}$ then N_β is V_β .

From this proposition, we see that, if $\beta \in \mathbf{Z}_{\geq 2}$, N_β is a low-discrepancy sequence. We also have the following theorem:

Theorem 2.1. Let $L, K \in \mathbf{N}$ and $K \leq L$. If $\beta = (L + \sqrt{L^2 + 4K})/2$, then N_β is a low-discrepancy sequence.

To prove this theorem, we provide several lemmas, propositions, and definitions. We use the following notation for periodic sequences:

$$\begin{aligned} &(a_1, a_2, \dots, \dot{a}_n, \dots, \dot{a}_{n+m}) \\ &= (a_1, a_2, \dots, a_n, a_{n+1}, \dots, a_{n+m}, a_n, a_{n+1}, \dots, a_{n+m}, \dots) \end{aligned}$$

Let $\beta \in \mathbf{R}_{>1}$.

Lemma 2.1. If $\zeta_\beta = (a_1, a_2, \dots, (a_m - 1))$, then $\{F_\beta(n)\}_{n=1}^\infty$ and $\{G_\beta(n)\}_{n=1}^\infty$ satisfy the following linear recurrent equations:

$$(2.1.F) \quad F_\beta(n+m) - \sum_{i=1}^m a_i F_\beta(n+m-i) = 0 \quad \text{for all } n \geq 1-m, n \neq 0$$

$$F_\beta(m) - \sum_{i=1}^m a_i F_\beta(m-i) + 1 = 0$$

$$(2.1.G) \quad G_\beta(n+m) - \sum_{i=1}^m a_i G_\beta(n+m-i) = 0 \quad \text{for all } n > 0.$$

Here we extend the definition of $F_\beta(n)$ to $F_\beta(-n) = 0$ ($n > 0$).

Proof. It is apparent from the definition of β -adic transformation that

$$(2.2.a) \quad a_1 = \begin{cases} [\beta], & \beta \notin \mathbf{Z} \\ \beta - 1, & \beta \in \mathbf{Z} \end{cases}$$

and

$$(2.2.b) \quad a_1 \geq \begin{cases} a_j, & j = 1, \dots, m-1 \\ a_m - 1, \end{cases}$$

hold. From Proposition 1.1, we have

$$\begin{aligned} X_\beta(n+m) = & \{(x, \omega_1) \mid x \in \{0, \dots, a_1 - 1\}, \omega_1 \in X_\beta(n+m-1)\} \\ & \cup \{(a_1, x, \omega_2) \mid x \in \{0, \dots, a_2 - 1\}, \omega_2 \in X_\beta(n+m-2)\} \\ & \vdots \\ & \cup \{(a_1, \dots, a_{m-1}, x, \omega_m) \mid x \in \{0, \dots, a_m - 1\}, \omega_m \in X_\beta(n)\} \end{aligned}$$

for all $n \geq 1$, and

$$\begin{aligned} X_\beta(0) &= \{(\dot{0})\} \\ X_\beta(1) &= \{(x, \dot{0}) \mid x \in \{1, \dots, a_1\}\} \\ X_\beta(2) &= \{(x, \omega_1) \mid x \in \{0, \dots, a_1 - 1\}, \omega_1 \in X_\beta(1)\} \\ &\quad \cup \{(a_1, x, \dot{0}) \mid x \in \{1, \dots, a_2\}\} \\ &\quad \vdots \\ X_\beta(m-1) &= \{(x, \omega_{m-2}) \mid x \in \{0, \dots, a_1 - 1\}, \omega_{m-2} \in X_\beta(m-2)\} \\ &\quad \cup \{(a_1, x, \omega_{m-3}) \mid x \in \{0, \dots, a_2 - 1\}, \omega_{m-3} \in X_\beta(m-3)\} \\ &\quad \vdots \\ &\quad \cup \{(a_1, \dots, a_{m-2}, x, \dot{0}) \mid x \in \{1, \dots, a_{m-1}\}\} \\ X_\beta(m) &= \{(x, \omega_{m-1}) \mid x \in \{0, \dots, a_1 - 1\}, \omega_{m-1} \in X_\beta(m-1)\} \\ &\quad \cup \{(a_1, x, \omega_{m-2}) \mid x \in \{0, \dots, a_2 - 1\}, \omega_{m-2} \in X_\beta(m-2)\} \\ &\quad \vdots \\ &\quad \cup \{(a_1, \dots, a_{m-1}, x, \dot{0}) \mid x \in \{1, \dots, a_m - 1\}\}. \end{aligned}$$

In the above expressions, we set $\{0, \dots, a_i - 1\} = \emptyset$ when $a_i = 0$. Remark $a_1, a_m \geq 1$. Then (2.1.F) holds. From Definition 2.1, (2.1.F), and

$$F_\beta(m) + F_\beta(0) = \sum_{i=1}^m a_i F_\beta(m-i),$$

we have

$$\begin{aligned} G_\beta(n+m) &= F_\beta(n+m) + F_\beta(n+m-1) + \dots + F_\beta(0) \\ &= a_1 F_\beta(n+m-1) + a_2 F_\beta(n+m-2) + \dots + a_m F_\beta(n) \\ &\quad + a_1 F_\beta(n+m-2) + a_2 F_\beta(n+m-3) + \dots + a_m F_\beta(n-1) \\ &\quad + \vdots \\ &\quad + a_1 F_\beta(m) + a_2 F_\beta(m-1) + \dots + a_m F_\beta(1) \\ &\quad + a_1 F_\beta(m-1) + a_2 F_\beta(m-2) + \dots + a_{m-1} F_\beta(0) \\ &\quad + a_1 F_\beta(m-2) + a_2 F_\beta(m-3) + \dots + a_{m-2} F_\beta(0) \\ &\quad + \vdots \\ &\quad + a_1 F_\beta(0) \\ &= a_1 G_\beta(n+m-1) + a_2 G_\beta(n+m-2) + \dots + a_m G_\beta(n). \end{aligned}$$

Thus (2.1.G) holds.

Definition 2.3. For $(k_1, k_2, \dots, k_n) \in X_\beta(n)$, define

$$d(k_1, k_2, \dots, k_n) = \min\{\max\{0, n-m\} \leq d \leq n \mid 1 \in \overline{B(\sigma^d(k_1, \dots, k_n))}\}.$$

Lemma 2.2. Let $(k_1, \dots, k_n) \in Y_\beta(n)$. When $(k_1, \dots, k_n) \in X_\beta(l)$ and $l < n$, we set $k_{l+1} = \dots = k_n = 0$. If $\zeta_\beta = (a_1, a_2, \dots, (a_m - 1))$, then

$$\lambda(B(k_1, \dots, k_n)) = \begin{cases} \frac{1}{\beta^d} \sum_{i=n-d+1}^m \frac{a_i}{\beta^i}, & \text{when } d > n-m \\ \frac{1}{\beta^n}, & \text{when } d = n-m \end{cases}$$

where $d = d(k_1, \dots, k_n)$ and λ is a one-dimensional Lebesgue measure.

Proof. From $\zeta_\beta = (a_1, a_2, \dots, (a_m - 1))$ we have

$$(2.3.a) \quad 1 - \sum_{i=1}^m \frac{a_i}{\beta^i} = 0$$

$$(2.3.b) \quad 1 - \sum_{i=1}^{lm} \frac{\zeta_\beta(i)}{\beta^i} = \frac{1}{\beta^{ml}},$$

where l is an arbitrary positive integer. If $\beta \in \mathbf{N}_{\geq 2}$, this lemma is trivial. We assume that $\beta \neq \mathbf{N}$. We prove the lemma by induction on n . Consider the case in which $n = 1$. From the definition of f_β , (2.2), and (2.3.a), we have

$$\lambda(B(0)) = \lambda(B(1)) = \cdots = \lambda(B(a_1 - 1)) = \frac{1}{\beta}$$

and

$$\lambda(B(a_1)) = \sum_{i=2}^m \frac{a_i}{\beta^i}.$$

This means that the lemma's statement holds when $n = 1$. We show that this statement holds for $(k_1, \dots, k_n, k_{n+1}) \in \cup_{i=1}^{n+1} X_\beta(i)$ under the induction hypothesis. For any $n \geq 1$ and $J \subset [0, 1)$,

$$(2.4) \quad f_\beta(f_\beta^{-n}(J)) = f_\beta^{-n+1}(J)$$

holds from f_β 's surjectivity. Consider the case in which $k_1 = 0, 1, \dots, a_1 - 1$, that is to say, the case in which $d = d(k_1, \dots, k_{n+1}) \geq 1$ and $d(k_2, \dots, k_{n+1}) = d - 1$. In this case, $f_\beta(B(k_1)) = [0, 1)$ holds; therefore, considering (2.4), we have

$$(2.5) \quad f_\beta(B(k_1, \dots, k_{n+1})) = B(k_2, \dots, k_{n+1})$$

and

$$(2.6) \quad \lambda(f_\beta(J)) = \beta \lambda(J)$$

for an arbitrary $J \subset B(k_1)$. From the induction hypothesis,

$$\lambda(B(k_2, \dots, k_{n+1})) = \begin{cases} \frac{1}{\beta^{d-1}} \sum_{i=n-d}^m \frac{a_i}{\beta^i}, & \text{when } d-1 > n-m \\ \frac{1}{\beta^n}, & \text{when } d-1 = n-m \end{cases}$$

holds. Therefore, from (2.5) and (2.6), this lemma's statement holds. When $d = 0$, the statement follows from (2.3.a) and (2.3.b).

For a sequence S , $S[N]$ denotes the point set consisting of the first N elements of S , and $S[N; M] = S[N + M] \setminus S[N]$.

Lemma 2.3. *For an arbitrary $(k_1, \dots, k_n) \in Y_\beta(n)$, we have*

$$\begin{aligned} & A(B(k_1, \dots, k_n); N_\beta[G_\beta(m + d + l)]) \\ &= \begin{cases} \sum_{i=1}^{m-n+d} a_{n-d+i} G_\beta(m + d + l - n - i) & \text{when } d > n - m \\ G_\beta(l) & \text{when } d = n - m \end{cases} \end{aligned}$$

where $d = d(k_1, \dots, k_n)$ and $l \in \mathbb{Z}_{\geq 0}$.

Proof. When $d = n - m$ holds, it is trivial. Assume that $d > n - m$. Let $K = (k_1, \dots, k_n)$. From Proposition 1.1,

$$\begin{aligned} & \{\omega \in \cup_{i=0}^{m+d+l} X_\beta(i) \mid \rho_\beta(\omega) \in B(k_1, \dots, k_n)\} \\ &= \{(K, x, \omega_1) \mid x \in \{0, \dots, a_{n-d+1} - 1\}, \omega_1 \in Y_\beta(m + d + l - n - 1)\} \\ &\cup \{(K, a_{n-d+1}, x, \omega_2) \mid x \in \{0, \dots, a_{n-d+2} - 1\}, \omega_2 \in Y_\beta(m + d + l - n - 2)\} \\ &\vdots \\ &\cup \{(K, a_{n-d+1}, \dots, a_{m-1}, x, \omega_{m-n+d}) \\ &\quad \mid x \in \{0, \dots, a_m - 1\}, \omega_{m-n+d} \in Y_\beta(l)\}, \end{aligned}$$

holds. In the above expressions, we set $\{0, \dots, a_i - 1\} = \emptyset$ when $a_i = 0$. Therefore, we have

$$\begin{aligned} & A(B(k_1, \dots, k_n); N_\beta[G_\beta(n + l)]) \\ &= \sum_{i=1}^{m-n+d-1} a_{n-d+i} G_\beta(m + d + l - i) + a_m G_\beta(n + l) \\ &= \sum_{i=1}^{m-n+d} a_{n-d+i} G_\beta(m + d + l - i). \end{aligned}$$

Proof of Theorem 2.1. From the conditions of the theorem,

$$(2.7) \quad \zeta_\beta = (\dot{L}, (K - 1))$$

holds. Let $\alpha = (L - \sqrt{L^2 + 4K})/2$. Then we have

$$(2.8.F) \quad F_\beta(n) = \begin{cases} 1, & n = 0 \\ \frac{1}{\beta - \alpha} (\beta^{n-1}(\beta^2 - 1) - \alpha^{n-1}(\alpha^2 - 1)), & n \geq 1 \end{cases}$$

$$(2.8.G) \quad G_\beta(n) = \begin{cases} 1, & n = 0 \\ \frac{1}{\beta - \alpha} (\beta^n(\beta + 1) - \alpha^n(\alpha + 1)), & n \geq 1 \end{cases}$$

from (2.7) and Lemma 2.1. Define $Z_\beta(n)$ and $H_\beta(n)$ as follows:

$$\begin{aligned} Z_\beta(n) &= \{\omega \in Y_\beta(n) \mid \omega(n) \neq L\} \\ H_\beta(n) &= \#Z_\beta(n) \end{aligned}$$

The following partitionings of $Y_\beta(n)$ and $Z_\beta(n)$ hold.

$$(2.9.Y) \quad \begin{aligned} Y_\beta(n + 1) &= \{(\omega, x) \mid x \in \{0, 1, \dots, K - 1\}, \omega \in Y_\beta(n)\} \\ &\cup \{(\omega, x) \mid x \in \{K, K + 1, \dots, L\}, \omega \in Z_\beta(n)\} \end{aligned}$$

$$(2.9.Z) \quad Z_\beta(n+1) = \{(\omega, x) \mid x \in \{0, 1, \dots, K-1\}, \omega \in Y_\beta(n)\} \\ \cup \{(\omega, x) \mid x \in \{K, K+1, \dots, L-1\}, \omega \in Z_\beta(n)\}$$

Then we have

$$(2.10) \quad \begin{aligned} H_\beta(n+1) &= KG_\beta(n) + (L-K)H_\beta(n) \\ G_\beta(n+1) &= KG_\beta(n) + (L-K-1)H_\beta(n). \end{aligned}$$

From (2.10) and Lemma 2.1, we have

$$H_\beta(n+2) - LH_\beta(n+1) - KH_\beta(n) = 0, \quad n \geq 1.$$

From the same discussion as in the proof of Lemma 2.3,

$$(2.11) \quad A(B(k_1, \dots, k_n); \rho_\beta(Z_\beta(2+d+l))) = \begin{cases} H_\beta(l), & d = n-2 \\ KH_\beta(l), & d = n-1 \\ H_\beta(l+2), & d = n \end{cases} \\ d = d(k_1, \dots, k_n)$$

holds for an arbitrary $(k_1, \dots, k_n) \in Y_\beta(n)$. Define

$$\Delta(B; P) = A(B; P) - M\lambda(B),$$

where B is an interval in $[0, 1)$ and $P = \{x_1, x_2, \dots, x_M\} \subset [0, 1)$. For any set of points P, S in $[0, 1)$, and any interval $B \subset [0, 1)$,

$$\Delta(B; P \cup S) = \Delta(B; P) + \Delta(B; S)$$

holds. Considering the order of N_β that we gave in Definition 2.2, we have

$$(2.12) \quad N_\beta[H_\beta(n)] = \rho_\beta(Z_\beta(n)).$$

From Lemma 2.2, Lemma 2.3, (2.8.G), (2.11) and (2.12), we have

$$(2.13) \quad \begin{aligned} &\Delta(B(k_1, \dots, k_n); N_\beta[G_\beta(2+d+l)]) \\ &= \begin{cases} \frac{\alpha+1}{\beta-\alpha} \left(\left(\frac{\alpha}{\beta} \right)^n - 1 \right) \alpha^l, & d = n-2 \\ \frac{K(\alpha+1)}{\beta-\alpha} \left(\left(\frac{\alpha}{\beta} \right)^{n+1} - 1 \right) \alpha^l, & d = n-1 \\ \frac{\alpha+1}{\beta-\alpha} \left(\left(\frac{\alpha}{\beta} \right)^n - 1 \right) \alpha^{l+2}, & d = n \end{cases} \end{aligned}$$

and

$$(2.14) \quad \begin{aligned} &\Delta(B(k_1, \dots, k_n); N_\beta[H_\beta(2+d+l)]) \\ &= \begin{cases} \frac{1}{\beta-\alpha} \left(\left(\frac{\alpha}{\beta} \right)^n - 1 \right) \alpha^{l+1}, & d = n-2 \\ \frac{K}{\beta-\alpha} \left(\left(\frac{\alpha}{\beta} \right)^{n+1} - 1 \right) \alpha^{l+1}, & d = n-1 \\ \frac{1}{\beta-\alpha} \left(\left(\frac{\alpha}{\beta} \right)^n - 1 \right) \alpha^{l+3}, & d = n \end{cases} \end{aligned}$$

where $(k_1, \dots, k_n) \in Y_\beta(n)$, $l \in \mathbb{Z}$ and $d = d(k_1, \dots, k_n)$. Define the truncating operator $r_k : X_\beta \rightarrow Y_\beta(k)$ as follows:

$$r_k(\omega) = \begin{cases} \omega, & \text{when } \omega \in X_\beta(j), j \leq k \\ (\omega(1), \dots, \omega(k)) & \text{otherwise} \end{cases}$$

For any $i, j \in \mathbb{Z}$ and any cylinder B of rank less than k ,

$$(2.15) \quad A(B; N_\beta[i; j]) = A(B; r_k(N_\beta[i; j]))$$

holds. Let $(k_1, \dots, k_n) \in Y_\beta(n)$, let $d = d(k_1, \dots, k_n)$, and let M be an arbitrary integer greater than $G_\beta(2 + d)$. Let l be an integer satisfying

$$G_\beta(2 + d + l) \leq M < G_\beta(2 + d + l + 1).$$

Applying partitioning (2.9.Y) and (2.9.Z) recursively for $Y_\beta(2 + d + l + 1)$, we obtain the following partitioning of $N_\beta[G_\beta(2 + d + l + 1)]$:

$$(2.16) \quad \begin{aligned} & N_\beta[G_\beta(2 + d + l + 1)] \\ &= N_\beta[G_\beta(2 + d + l)] \\ & \quad \cup N_\beta[G_\beta(2 + d + l); G_\beta(2 + d + l)] \\ & \quad \vdots \\ & \quad \cup N_\beta[(K - 1)G_\beta(2 + d + l); G_\beta(2 + d + l)] \\ & \quad \cup N_\beta[KG_\beta(2 + d + l); H_\beta(2 + d + l)] \\ & \quad \vdots \\ & \quad \cup N_\beta[KG_\beta(2 + d + l) + (L - K - 1)H_\beta(2 + d + l); H_\beta(2 + d + l)] \\ & \quad \cup N_\beta[KG_\beta(2 + d + l) + (L - K)H_\beta(2 + d + l); G_\beta(2 + d + l - 1)] \\ & \quad \vdots \\ & \quad \cup N_\beta[KG_\beta(2 + d + l) + (L - K)H_\beta(2 + d + l) + KG_\beta(2 + d + l - 1) \\ & \quad \quad ; H_\beta(2 + d + l - 1)] \\ & \quad \cup \\ & \quad \vdots \end{aligned}$$

Partition $N_\beta[M]$ in the same way as (2.16); then, from (2.15), the additivity of Δ ,

(2.9.Y), (2.9.Z), and the order we induced to N_β , we have

$$\begin{aligned}
 & \Delta(B; N_\beta[M]) \\
 & \leq K |\Delta(B; N_\beta[G_\beta(2+d+l)])| + (L-K) |\Delta(B; N_\beta[H_\beta(2+d+l)])| \\
 & \quad + K |\Delta(B; N_\beta[G_\beta(1+d+l)])| + (L-K-1) |\Delta(B; N_\beta[H_\beta(1+d+l)])| \\
 & \quad + K |\Delta(B; N_\beta[G_\beta(d+l)])| + (L-K-1) |\Delta(B; N_\beta[H_\beta(d+l)])| \\
 & \quad \vdots \\
 (2.17) \quad & + K |\Delta(B; N_\beta[G_\beta(2+d+1)])| + (L-K-1) |\Delta(B; N_\beta[H_\beta(2+d+1)])| \\
 & \quad + K |\Delta(B; N_\beta[G_\beta(2+d)])| + (L-K) |\Delta(B; N_\beta[H_\beta(2+d)])| \\
 & \leq K \sum_{i=0}^l |\Delta(B; N_\beta[G_\beta(2+d+i)])| \\
 & \quad + (L-K) \sum_{i=0}^l |\Delta(B; N_\beta[H_\beta(2+d+i)])|
 \end{aligned}$$

where $B = B(k_1, \dots, k_n)$. From (2.13), (2.14), (2.17) and the fact that $|\alpha| < 1 < |\beta|$, there exists a constant C_1 that satisfies the following inequality (2.18) for any cylinder $B(k_1, \dots, k_n)$ of any rank n and any integer $M > G_\beta(2+d)$.

$$(2.18) \quad |\Delta(B(k_1, \dots, k_n); N_\beta[M])| < C_1$$

Choose an arbitrary $u \in [0, 1)$. Let $M \in \mathbb{N}$ and l be an integer that satisfies

$$G_\beta(l) \leq M < G_\beta(l+1).$$

Let $B(u_1, \dots, u_l)$ be a cylinder of rank l that satisfies $u \in B(u_1, \dots, u_l)$. Then we have

$$\begin{aligned}
 (2.19) \quad [0, u) &= B_{t_1} \sqcup B_{t_2} \sqcup \dots \sqcup B_{t_k} \sqcup R \\
 &0 \leq t_1 < t_2 < \dots < t_k = l
 \end{aligned}$$

where B_{t_i} is a cylinder of rank t_i and $\lambda(R) < \beta^{-l}$. From (2.8.G), there exist constants C_2 and C_3 that satisfy $l < C_2 \log M$ and $M\beta^{-l} < C_3$. Then, from (2.18) and (2.19), we have

$$|\Delta([0, u); N_\beta[M])| < C_1 C_2 \log M + C_3.$$

The theorem follows from this.

References

1. Harald Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, 1992.
2. W. Parry, *On the β -expansions of real numbers*, Acta Math. Acad. Sci. Hungar **11** (1960), 401–416.
3. A. Rény, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar **8** (1957), 477–493.
4. W. M. Schmidt, *Irregularities of distribution VII*, Acta Arith. **21** (1972), 45–50.
5. Fritz Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory*, Oxford science publications, 1995.
6. Y. Takahashi, *Kukanrikigakukei no Kaosu to Shuhkiten*, (in Japanese), Tokyo Metropolitan University Seminar Report.

IBM Research, Tokyo Research Laboratory 1623-14, Shimotsuruma, Yamato-shi,
Kanagawa-ken 242, Japan

E-mail address: ninomiya@trl.ibm.co.jp